On numerical approximation of the Hamilton-Jacobi-transport system arising in high frequency approximations

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Abstract

In the present article, we study the numerical approximation of a system of Hamilton-Jacobi and transport equations arising in geometrical optics. We consider a semi-Lagrangian scheme. We prove the well posedness of the discrete problem and the convergence of the approximated solution toward the viscosity-measure valued solution of the exact problem.

Key words. Hamilton-Jacobi equation; eikonal equation; transport equation; viscosity solutions; measure solutions; semiconcavity; numerical approximations; OSL condition.

AMS subject classification: 65M12; 35F25; 35R05;49L25;

1 Introduction

In this article, we consider the following system

$$\partial_t u + H(x,t,\nabla u) = 0\,, \qquad \text{in } \mathbb{R}^d \times (0,T], \tag{1.1}$$

$$\partial_t m + \nabla \cdot (a(x,\nabla u)\, m) = 0\,, \qquad \text{in } \mathbb{R}^d \times (0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \quad m(0) = m_0\,, \qquad \text{in } \mathbb{R}^d\,, \tag{1.3}$$

of an Hamilton-Jacobi type equation (HJ) and a continuity equation (CE) describing the transport of the conserved measure m_0. Even if the vector field a can be smooth (in the simplest and of reference case, $a(x,p) = p$), the scalar function u as a solution of (1.1) is intended in the viscosity sense. Therefore, ∇u is at most L^∞, even for smooth initial data u_0, and the regularity that we can expect for the vector field $a(\cdot,\nabla u)$ is the very weak regularity

$$a(\cdot\,,\nabla u(\cdot,\cdot)) \in (L^\infty(\mathbb{R}^d \times [0,T]))^d\,. \tag{1.4}$$

As a consequence, despite the fact that the (HJ) equation can be solved independently from (CE), system (1.1)-(1.2) leads to some interesting mathematical issues as well as numerical challenges.

More specifically, in order to obtain global solutions, a well adapted notion of solution for (1.2) under hypothesis (1.4) is the notion of *measure solutions* introduced by Poupaud-Rascle [16]. Indeed,

the latter perfectly makes sense if the vector field $a(\cdot, \nabla u)$ satisfies a *one-sided Lipschitz condition* (OSL), (see (2.3) below), and this condition can be obtained by the semiconcavity of the viscosity solution of (1.1), at least in the reference case $a(x,p) = p$, or in the one dimensional case for a class of functions a.

Existence and uniqueness results for problem (1.1)-(1.2)-(1.3) in the framework of viscosity-measure solutions have been given for example in [2, 19] and in the different framework of the one dimensional viscosity-duality solutions in [12]. Therefore, our goal here is not to refine these previous results but to construct consistent numerical approximations.

The considered scheme is based on a semi-lagrangian discretization of (1.1)-(1.2) for the time approximation coupled with a finite element discretization for the space variable. It requires a convex hamiltonian H. Since the semiconcavity of the initial data u_0 is conserved by the scheme, it is sufficient to require only a weak (OSL) condition at the discrete level (automatically satisfied in the reference case $a(x,p) = p$) without semiconcavity requirement for the viscosity solution u.

The stability of the Filippov characteristics and of the corresponding measure solution of the transport equation (CE) will be the key tool to prove the convergence of the numerical schemes. As a by-product, we obtain of course a new existence and uniqueness result for the viscosity-measure solution of (1.1)-(1.2)-(1.3).

Regarding the applications of our numerical approximations of (1.1)-(1.2)-(1.3), it is worth to recall that these types of systems arise for example in the semi-classical limit for the Schrödinger equation [4] and of the spinless Bethe-Salpeter equation (the relativistic Schrödinger equation, [3]), or in the high frequency approximation of the Helmholtz equation. In these cases, the hamiltonian H and the vector field a take the forms

$$H(x,t,p) = \frac{1}{2}|p|^2 + V(x,t)\,, \quad a(x,p) = p\,, \quad (\textit{Schrödinger and Helmholtz equation}), \tag{1.5}$$

$$H(x,t,p) = \left(\frac{|p|^2}{2} + 1\right)^{1/2} + V(x,t)\,, \quad a(x,p) = p\left(\frac{|p|^2}{2} + 1\right)^{-1/2}, \quad (\textit{Bethe-Salpeter equation}), \tag{1.6}$$

where $V(x,t)$ is the potential (see [2, 12, 19] for a rapid derivation of (1.5) and (1.6)).

Coupling between first or second order Hamilton-Jacobi equations and transport equations has been also recently considered in the framework of Mean Field Games theory [13]. In this case however, the system is given by a backward Hamilton-Jacobi equation and a forward transport equation with initial-terminal conditions and coupling terms in both equations. The resulting system turns out to be completely different from the one considered here.

The paper is organized as follows. Section 2 is devoted to the preliminary definitions and known results concerning system (1.1)–(1.2)–(1.3). In Section 3 we construct a semi-lagrangian scheme for the time discretization of system (1.1)–(1.2)–(1.3) and we prove its convergence. The corresponding fully discrete scheme is given and analyzed in Section 4. Finally, the appendix is devoted to the proof of some technical lemmas for the reader convenience.

Throughout the paper, we will denote by $C_0^0(\mathbb{R}^d)$ (resp. $C_c^0(\mathbb{R}^d)$) the space of continuous functions

which tend to 0 at infinity (resp. with compact support); by ρ_ε a standard mollifier, i.e. $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, $\rho \in C_c^\infty(\mathbb{R}^d)$, $\rho \geq 0$ and $\int_{\mathbb{R}^d} \rho(x)dx = 1$; by $*$ the convolution with respect the space variable and by C any numerical constant that can vary from line to line in the computations.

2 Preliminaries : the viscosity-measure solutions

As mentioned in the introduction, a solution of (1.1)–(1.2) is intended in the viscosity sense for (1.1), while in the sense of Poupaud-Rascle [16] for (1.2). Concerning the definition of viscosity solutions, we refer to the pioneering articles [6, 5]. Here, for the reader's convenience, we recall the usual assumptions on H and the consequent general existence and uniqueness result for (1.1) that we shall use in the sequel.

Let us define $Q_T := \mathbb{R}^d \times [0,T]$, $T > 0$, $d \geq 1$, and let the hamiltonian H satisfy the following:
($\mathbf{H}_1$) H is uniformly continuous on $Q_T \times B(0,R)$ for any $R > 0$;
($\mathbf{H}_2$) $H(x,t,0)$ is uniformly bounded : $\sup_{Q_T} |H(x,t,0)| \equiv M < +\infty$;
($\mathbf{H}_3$) there exists $\eta > 0$ s.t. : $|H(x,t,p) - H(y,t,p)| \leq \eta\,(1 + |p|)|x - y|\,,\ \forall\, t \in [0,T]\,,\ \forall\, x, y, p \in \mathbb{R}^d$.

Theorem 2.1 ([6])**.** *Under hypothesis* ($\mathbf{H}_1$), ($\mathbf{H}_2$) *and* ($\mathbf{H}_3$), *if in addition the initial data* u_0 *belongs to* $(W^{1,\infty} \cap BUC)(\mathbb{R}^d)$, *there exists a unique viscosity solution* $u \in (W^{1,\infty} \cap BUC)(Q_T)$ *of* (1.1), (1.3).

As stated in Theorem 2.1, the expected regularity for ∇u is $L^\infty(Q_T)$. Therefore, the characteristics $X(t;x)$ associated to the conservative transport equation (1.2) cannot be defined as classical or distributional solutions of the system below

$$\begin{cases} \partial_t X(t;x) = a(X(t;x), \nabla u(X(t;x),t)) \\ X(0;x) = x\,, \qquad x \in \mathbb{R}^d\,, \end{cases} \tag{2.1}$$

but have to be understood in a generalized sense. Once the flow $X(t;x)$ is uniquely defined and continuous on $[0,T] \times \mathbb{R}^d$, the natural definition of a solution of the conservation law (1.2) with a given initial data m_0 belonging to the space of bounded measures $\mathcal{M}_1(\mathbb{R}^d)$ is, see[16],

$$m(t) = X(t\,;\cdot)\,\#\,m_0\,. \tag{2.2}$$

Equation (2.2) means that the measure solution $m(t)$ is the image (or the push-forward) of m_0 by the flow $X(t\,;\cdot)$, i.e. for any Borel set $B \subset \mathbb{R}^d$,

$$m(t)(B) = m_0(X(t\,;\cdot)^{-1}(B)) = m_0(\{x \in \mathbb{R}^d : X(t;x) \in B\})\,,$$

or equivalently

$$\langle m(t), \phi\rangle = \langle m_0, \phi(X(t;\cdot))\rangle\,, \qquad \text{for any } \phi \in C_0^0(\mathbb{R}^d)\,.$$

It turns out that the definition of Filippov characteristics is well suited and that with it, (2.2) perfectly make sense as soon as the characteristics are unique. A definition of Filippov characteristics solution of (2.1) equivalent to the original one and easier to handle is given in [16], as follows:

Definition 2.2 (Filippov characteristics)**.** *For any given $x \in \mathbb{R}^d$ and $T > 0$, a generalized solution in the sense of Filippov of system* (2.1) *is a continuous map $X(\cdot\,; x) : [0,T] \mapsto \mathbb{R}^d$ which satisfies*

$$\{\underline{\mathcal{M}}(a(\cdot, \nabla u(\cdot, t)) \cdot v)\}(X(t;x)) \le \partial_t X(t;x) \cdot v \le \{\overline{\mathcal{M}}(a(\cdot, \nabla u(\cdot, t)) \cdot v)\}(X(t;x)) , \quad a.e.\ t \in [0,T] ,$$

for any $v \in \mathbb{R}^d$, where

$$\{\underline{\mathcal{M}}(a(\cdot, \nabla u(\cdot, t)) \cdot v)\}(z) := \sup_{r>0} \left(\operatorname*{ess\,inf}_{y \in B(z,r)} a(y, \nabla u(y,t)) \cdot v \right)$$
$$\{\overline{\mathcal{M}}(a(\cdot, \nabla u(\cdot, t)) \cdot v)\}(z) := \inf_{r>0} \left(\operatorname*{ess\,sup}_{y \in B(z,r)} a(y, \nabla u(y,t)) \cdot v \right) .$$

Then, assuming

($\mathbf{A}_1$) $|a| \in L^\infty_x(\mathbb{R}^d\,; L^\infty_{loc,\,p}(\mathbb{R}^d))$,

the vector field $a(\cdot, \nabla u)$ satisfies (1.4) and the Filippov characteristics of system (2.1) exist. Moreover, if the following *one-sided Lipschitz condition* (OSL) holds true

$$(a(x, \nabla u(x,t)) - a(y, \nabla u(y,t))) \cdot (x - y) \le \gamma(t)|x - y|^2 \quad \text{a.e. } t \in [0,T],\ x, y \in \mathbb{R}^d \tag{2.3}$$

for a function $\gamma \in L^1([0,T])$, these characteristics are unique, the flow $(t,x) \mapsto X(t;x)$ is continuous on $[0,T] \times \mathbb{R}^d$, the map from $\mathbb{R}^d$ to $\mathbb{R}^d$, $x \mapsto X(t;x)$ is onto; the following existence and uniqueness result of a viscosity-measure solution of (1.1)-(1.2)-(1.3) can be stated:

Theorem 2.3. *Let $u_0 \in (W^{1,\infty} \cap BUC)(\mathbb{R}^d)$ and $m_0 \in \mathcal{M}_1(\mathbb{R}^d)$. Assume* ($\mathbf{H}_1$)-($\mathbf{H}_2$)-($\mathbf{H}_3$) *and* ($\mathbf{A}_1$)*. Then, if the viscosity solution u satisfies* (2.3)*, there exists a unique measure solution $m \in C^0([0,T]; \mathcal{M}_1(\mathbb{R}^d) - weak\,*)$ of* (1.2)-(1.3) *given by* (2.2)*.*

In the reference case $a(x,p) = p$, as in (1.5), requiring the (OSL) condition (2.3) is equivalent to require that the viscosity solution u be semiconcave with respect to x, i.e.

$$u(x + y, t) - 2\,u(x,t) + u(x - y, t) \le \beta(t)\,|y|^2 , \tag{2.4}$$

for all $x, y \in \mathbb{R}^d$, $t \in [0,T]$ and for some $\beta \in L^1([0,T])$. The semiconcavity property (2.4) often characterizes the viscosity solution of Hamilton-Jacobi equations arising in control problems. However, for general vector fields a, it does not imply the necessary condition (2.3). Indeed, for $u_\varepsilon = u * \rho_\varepsilon$, where ρ_ε is the mollifier introduced above, the semiconcavity (2.4) implies: $\nu^T\, D^2 u^\varepsilon(x,t)\, \nu \le \beta(t)$, for all $\nu \in \mathbb{R}^d$ s.t. $|\nu| = 1$ and for all $(x,t) \in Q_T$, (see [7]). Moreover, $\nabla u^\varepsilon(x,t) \to \nabla u(x,t)$ as $\varepsilon \to 0$, for a.a. $(x,t) \in Q_T$. Owing to this convergence property, it is sufficient to obtain (2.3) for u^ε, whenever a is at least continuous w.r.t. p. Assuming in addition that a is differentiable w.r.t. p and satisfies a one sided Lipschitz condition w.r.t. x locally uniformly in p, we have

$$a(x,p) - a(y,q) = \int_0^1 D_p a(x, q + s(p - q))(p - q)\, ds + a(x,q) - a(y,q) ,$$

and the estimate

$$
\begin{aligned}
&(a(x,\nabla u^\varepsilon(x,t)) - a(y,\nabla u^\varepsilon(y,t)))\cdot(x-y) \le \\
&\qquad \int_0^1 ds \int_0^1 d\sigma\,(x-y)^T\,\Gamma(s)\,D_x^2 u^\varepsilon(y+\sigma(x-y),t)(x-y) + C\,|x-y|^2\,,
\end{aligned}
\tag{2.5}
$$

with $\Gamma(s) = D_p a(x, \nabla u^\varepsilon(y,t) + s(\nabla u^\varepsilon(x,t) - \nabla u^\varepsilon(y,t)))$. Therefore, the (OSL) condition follows easily from (2.4), if either $D_p a(x,p) = I$, as expected, or $d = 1$ and $\partial_p a(x,p)$ is non negative and upper bounded, as for (1.6). Apart from these two cases (already considered in [2]), the (OSL) condition is an assumption and has to be verified for the specific model at hand.

We conclude this section by listing the properties that the measure solution (2.2) satisfies, see [16] :
(i) *monotonicity* : $m_0 \ge \nu_0 \Rightarrow X(t\,;\cdot)\,\#\,m_0 \ge X(t\,;\cdot)\,\#\,\nu_0$;
(ii) *mass conservation* : $m(t)(\mathbb{R}^d) = m_0(\mathbb{R}^d)$;
(iii) *contraction property* : $|m(t)|(B) \le |m_0|(B)$, for any Borel set $B \subseteq \mathbb{R}^d$;
(iv) *semi-group property* : $X(t-s\,;\cdot)\,\#\,(X(s\,;\cdot)\,\#\,m_0) = X(t\,;\cdot)\,\#\,m_0$;
(v) *uniform compactness at infinity* : $\forall\,\varepsilon > 0$ there exists $R > 0$ s.t. $|m(t)|(\mathbb{R}^d \backslash B_R(0)) \le \varepsilon$, $\forall\,t \in [0,T]$.
It is worth to underline that the *uniform compactness at infinity* property (v) is fundamental to prove the convergence of approximate measure solutions toward the exact one in $C^0([0,T];\mathcal{M}_1(\mathbb{R}^d)\,w-*)$.

3 The semi-lagrangian scheme

This section is devoted to the construction and the convergence analysis of a semi-lagrangian scheme for the time discretization of system (1.1)-(1.2) over the time interval $[0,T]$. For an introduction to this class of schemes we refer to [1, Appendix B] and [8, 9, 10, 11]. To proceed, we need to refine the previous hypothesis on the hamiltonian H and to add assumptions on the growth of H w.r.t. p. Therefore, let us assume in this section $(\mathbf{H}_1)$, $(\mathbf{H}_3)$ and

$(\mathbf{H}_2')$ for any $R > 0$, $\sup_{Q_T \times \overline{B}(0,R)} |H(x,t,p)| \equiv M(R) < +\infty$;

$(\mathbf{H}_4)$ H is convex in p and either linear at infinity (i) or superlinear (ii):

(i) $\exists\; K > 0$ s.t. $\lim\limits_{|p|\to\infty} \frac{H(x,t,p)}{|p|} = K$, uniformly in $(x,t) \in Q_T$, and there exists a positive function $\alpha \in L^\infty(\mathbb{R}^d)$ such that $|H(x,t,p) - H(y,t,p)| \le \alpha(p)|x-y|$ uniformly on $Q_T \times \mathbb{R}^d$;

(ii) $\lim\limits_{|p|\to\infty} \inf\limits_{Q_T} \frac{H(x,t,p)}{|p|} = +\infty$.

Remark 3.1 It is worth noticing that the hamiltonians (1.5) and (1.6) satisfy all the required assumptions as soon as the potential V is uniformly continuous and bounded over Q_T and Lipschitz continuous in x uniformly in t. The growth assumption $(\mathbf{H}_4)$-(i) can be relaxed to include also hamiltonians that are not uniformly positive at infinity. But we leave this generalization to the reader.

3.1 The semi-lagrangian scheme for the (HJ) equation

Let $N \in \mathbb{N}$ be fixed and let $h = T\,N^{-1}$ be the associated time step used in the semi-discrete scheme to be defined. Since the hamiltonian H is convex in p and continuous (in fact lower semi-continuity would be sufficient [17]), then $H(x,t,p) = H^{**}(x,t,p)$, where H^* is the Legendre transform of H with respect to p, i.e. $H^*(x,t,\xi) = \sup_{p\in\mathbb{R}^d}\{\xi \cdot p - H(x,t,p)\}$, and the (HJ) equation can be written as

$$\partial_t u(x,t) = \inf_{\xi\in\mathbb{R}^d}\{-\xi \cdot \nabla u(x,t) + H^*(x,t,\xi)\}. \tag{3.1}$$

Next, plugging the first order forward finite difference for the approximation of $\partial_t u$

$$\partial_t u(x,t) \sim \frac{u(x,t+h) - u(x,t)}{h}$$

and the first order approximation of the directional derivative $-\xi \cdot \nabla u$

$$-\xi \cdot \nabla u(x,t) \sim \frac{u(x-\xi h,t) - u(x,t)}{h}\,, \tag{3.2}$$

into (3.1), one easily obtains the following first order approximation for $u(x,t+h)$

$$u(x,t+h) \sim \inf_{\xi\in\mathbb{R}^d}\{u(x-\xi\,h,t) + h\,H^*(x,t,\xi)\}\ .$$

Let us observe that whenever the exact solution u of (HJ) is semiconcave in x, then it possesses one-sided directional derivatives at any x and in any direction, i.e. the limit as $h \searrow 0$ of the right hand side of (3.2) always exists. Moreover, if u is differentiable w.r.t. x, the previous limit coincides with the left hand side of (3.2). In other word, the approximation (3.2) is consistent.

For the semi-discrete scheme, it is sufficient to inductively define the approximation $u^n = u^n(x)$ of the exact solution u of (HJ) at time $t^n = n\,h$ for $n = 0,\ldots,N$, by

$$u^{n+1}(x) = \inf_{\xi\in\mathbb{R}^d}\{u^n(x-\xi h) + h\,H^*(x,t^n,\xi)\}\ , \quad x \in \mathbb{R}^d\,, \tag{3.3}$$

with $u^0 = u_0$, the initial data for (HJ) in (1.3). Next, we need to show that u^n as given by (3.3) shares the properties of u_0, in the same way as the exact viscosity solution u does. These are well established facts in the context of the approximation of functions by the inf-convolution operator (see for example [7]). However, the difficulties here are on the one hand to show that the properties of the initial data u_0 are propagated to u^n uniformly with respect to n and h, and on the other hand to handle the dependency of H on (x,t).

Let us define $Q_h := \mathbb{R}^d \times \{0,\ldots,N\}$ and the set of the arguments associated to the infimum in (3.3)

$$A^n(x) := \arg\inf_{\xi\in\mathbb{R}^d}\{u^n(x-\xi h) + h\,H^*(x,t^n,\xi)\}\ , \quad (x,n) \in Q_h\,. \tag{3.4}$$

Lemma 3.2 (Properties of H^*)**.** *Let H satisfy* $(\mathbf{H}_1)$*,* $(\mathbf{H}_2')$ *and* $(\mathbf{H}_3)$*. If in addition H satisfies* $(\mathbf{H}_4)$*-(i), then :*

(a) $H^*(x,t,\xi) = +\infty$ *if* $|\xi| > K$*, for any* $(x,t) \in Q_T$ *,* $H^*(\cdot,\cdot,0) \in L^\infty(Q_T)$ *and* H^* *is Lipschitz continuous in* x *uniformly in* $(\xi, t) \in \overline{B}(0,K) \times [0,T]$*.*

On the other hand, if in addition H *satisfies* $(\mathbf{H}_4)$*-(ii), then :*

(b) H^* *also satisfies* $(\mathbf{H}_4)$*-(ii), for any* $r > 0$ *there exists* $R = R(r) > 0$ *such that*

$$H^*(x,t,\xi) = \max_{p \in \overline{B}(0,R)} \{p \cdot \xi - H(x,t,p)\}, \qquad \forall\ (x,t,\xi) \in Q_T \times \overline{B}(0,r), \tag{3.5}$$

$H^* \in L^\infty(Q_T \times \overline{B}(0,r))$ *and* H^* *is Lipschitz continuous in* x *uniformly in* $(\xi,t) \in \overline{B}(0,r) \times [0,T]$*.*

The proof of the Lemma above is given in the Appendix. Let us just point out here that we have chosen to take the hamiltonian H to be Lipschitz continuous in x uniformly in p and t when H is linear in p for $|p| \to +\infty$, in order to obtain the Lipschitz continuity of H^* w.r.t. x. Indeed, in this case the supremum defining H^* is not a priori reached. Therefore we cannot conclude as in case $(\mathbf{H}_4)$-(ii).

Lemma 3.3 (Properties of u^n)**.** *Let* $u_0 \in W^{1,\infty}(\mathbb{R}^d)$*. Then, under hypothesis* $(\mathbf{H}_1)$*,* $(\mathbf{H}_2')$*,* $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$*,* u^n *is well defined over* Q_h *by* (3.3)*, i.e.* $A^n(x)$ *is a non empty bounded set of* $\mathbb{R}^d$ *uniformly in* $(x,n) \in Q_h$ *and the infimum is a minimum. Moreover,* $u^n \in W^{1,\infty}(\mathbb{R}^d)$ *and* $\|u^n\|_{W^{1,\infty}(\mathbb{R}^d)}$ *is bounded uniformly in* n *and, assuming* $H^* \in L^\infty(Q_T \times \overline{B}(0,K))$ *when the growth condition* $(\mathbf{H}_4)$*-(i) is satisfied,* u^n *is Lipschitz continuous with respect to* n *uniformly in* x*, i.e. there exist three positive constants* C_0*,* C_1 *and* C_2 *independent of* h *and* n*, s.t.*

$$\|u^n\|_{L^\infty(\mathbb{R}^d)} \le C_0\,, \quad \|\nabla u^n\|_{L^\infty(\mathbb{R}^d)} \le C_1 \quad \text{and} \quad |u^n(x) - u^m(x)| \le C_2|n-m|\,h\,, \tag{3.6}$$

for all $n, m = 0, \cdots, N$*, and* $x \in \mathbb{R}^d$*.*

Proof. We suppose that $u^n \in W^{1,\infty}(\mathbb{R}^d)$ with

$$\|u^n\|_{L^\infty(\mathbb{R}^d)} \le C_0^n \quad \text{and} \quad \|\nabla u^n\|_{L^\infty(\mathbb{R}^d)} \le C_1^n\,,$$

where C_0^n and C_1^n are independent of h. Then, the proof will follow by induction on n.

It follows immediately by Lemma 3.2 that u^{n+1} is upper bounded since

$$u^{n+1}(x) \le u^n(x) + h\,H^*(x,t^n,0) \le C_0^n + h\,\|H^*(\cdot,\cdot,0)\|_{L^\infty(Q_T)}, \quad \forall x \in \mathbb{R}^d\,.$$

Moreover, u^{n+1} is obviously lower bounded since u^n is bounded and $H^*(x,t,\xi) \ge -M$, for any $(x,t,\xi) \in Q_T \times \mathbb{R}^d$, where $M \equiv \sup_{Q_T} |H(x,t,0)|$, so that

$$\|u^{n+1}\|_{L^\infty(\mathbb{R}^d)} \le C_0^n + h \max\{M,\ \|H^*(\cdot,\cdot,0)\|_{L^\infty(Q_T)}\}\,.$$

Next, if H satisfies $(\mathbf{H}_4)$-(i), $A^n(x) \subset \overline{B}(0,K)$ for any $x \in \mathbb{R}^d$ and the infimum in (3.3) is attained due to the continuity of $u^n(x - \xi h) + h\,H^*(x,t^n,\xi)$ w.r.t. ξ.

On the other hand, if H satisfies $(\mathbf{H}_4)$-(ii), since u^n and $H^*(x,t,0)$ are bounded and H^* is superlinear, there exists $R^n > 0$ (increasing w.r.t. n but upper bounded uniformly in n and h) s.t. $A^n(x) \subset \overline{B}(0,R^n)$ for any $x \in \mathbb{R}^d$ and again the infimum in (3.3) is attained.

Let us prove now that u^{n+1} is Lipschitz continuous. We have for any $x, y \in \mathbb{R}^d$ and for $\alpha^n(x) \in A^n(x)$

$$u^{n+1}(x) = u^n(x - h\,\alpha^n(x)) + h\,H^*(x,t^n,\alpha^n(x))$$

and

$$u^{n+1}(y) \le u^n(y - h\,\alpha^n(x)) + h\,H^*(y,t^n,\alpha^n(x))\,.$$

Therefore,

$$u^{n+1}(y) - u^{n+1}(x) \le [\|\nabla u^n\|_\infty + h\,L_{H^*}]\,|x-y| \le [C_1^n + h\,L_{H^*}]\,|x-y|\,,$$

where L_{H^*} is the Lipschitz constant of H^* w.r.t. x, and the statement follows exchanging the role of x and y. Consequently, $u^{n+1} \in W^{1,\infty}(\mathbb{R}^d)$.

It remains to prove that u^n is Lipschitz continuous with respect to n. As before, we have for any $x \in \mathbb{R}^d$ and the corresponding argument $\alpha^n(x) \in A^n(x)$

$$\begin{aligned}|u^{n+1}(x) - u^n(x)| &= |u^n(x - h\,\alpha^n(x)) - u^n(x + h\,H^*(x,t^n,\alpha^n(x))| \\ &\le \|\nabla u^n\|_\infty\, h\,|\alpha^n(x)| + h|H^*(x,t^n,\alpha^n(x))| \le C_1^n h\,|A^n| + h|H^*(x,t^n,\alpha^n(x))|\,,\end{aligned}$$

and the proof is completed, thanks to the uniform boundedness of A^n and H^*. $\square$

Let us observe that when the hamiltonian H grows linearly at infinity, i.e. when the growth condition $(\mathbf{H}_4)$-(i) is satisfied, then H^* is not necessarily upper bounded. Therefore, the additional hypothesis $H^* \in L^\infty(Q_T \times \overline{B}(0,K))$ in Lemma 3.3 is necessary. It is easily seen that this additional hypothesis is satisfied by the hamiltonian (1.6).

Finally, in order to obtain semiconcavity, we need the following semiconcavity hypothesis on H^*.

$(\mathbf{H}_5)$ $H^* = H^*(x,t,\xi)$ is semiconcave in x uniformly in t and ξ, with semiconcavity constant $C_{conc}^{H^*}$.

Lemma 3.4 (Semiconcavity)**.** *Let $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ be semiconcave. Then, under hypothesis $(\mathbf{H}_1)$, $(\mathbf{H}_2')$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$, u^n is also semiconcave uniformly in h and n.*

Proof. We proceed as in Lemma 3.3 supposing that u^n is semiconcave with semiconcavity constant C_{conc}^n . Then, for any $x \in \mathbb{R}^d$, any corresponding argument $\alpha^n(x) \in A^n(x)$ and any $y \in \mathbb{R}^d$, we have that

$$\begin{aligned}&u^{n+1}(x+y) - 2u^{n+1}(x) + u^{n+1}(x-y) \\ &\quad\le u^n(x+y-h\,\alpha^n(x)) - 2\,u^n(x-h\,\alpha^n(x)) + u^n(x-y-h\,\alpha^n(x)) \\ &\qquad + h\,[H^*(x+y,t^n,\alpha^n(x)) - 2\,H^*(x,t^n,\alpha^n(x)) + H^*(x-y,t^n,\alpha^n(x))] \\ &\quad\le \left[C_{conc}^n + h\,C_{conc}^{H^*}\right]|y|^2\,.\end{aligned}$$

Hence, u^{n+1} is semiconcave with $C^{n+1}_{conc} \leq C^n_{conc} + h\, C^{H^*}_{conc}$, and by induction u^n is semiconcave for all $n = 1, \ldots, N$ with : $C^n_{conc} \leq C^{u_0}_{conc} + T\, C^{H^*}_{conc}$.

□

Remark 3.5 Assumption ($\mathbf{H}_5$) is necessary since the hamiltonian H depends on x and t. It can be satisfied under regularity hypothesis on H. In the case of the hamiltonians (1.5) and (1.6), ($\mathbf{H}_5$) is satisfied if the potential V is convex in x uniformly in t.

3.2 The semi-lagrangian scheme for the (CE) equation

Now we can proceed to the construction of a semi-discrete scheme for the transport equation (1.2), replacing the time continuous flow $X(t;x)$ with a time discrete one. Set $u^n_\varepsilon = u^n * \rho_\varepsilon$ and let us consider the following implicit backward Euler scheme

$$\begin{cases} X^{n+1} = X^n + h\, a(X^{n+1}, \nabla u^{n+1}_\varepsilon(X^{n+1}))\,, & n = 0, \cdots, N-1, \\ X^0 = x\,, \qquad x \in \mathbb{R}^d\,. \end{cases} \tag{3.7}$$

Next, given the initial measure m_0 and replacing (2.1) with (3.7), we define the semi-discrete approximation of the measure solution (2.2) as

$$m^n = X^n(\cdot) \,\#\, m_0\,, \qquad \text{for } n = 0, \ldots, N\,, \tag{3.8}$$

i.e. for any Borel set $B \subset \mathbb{R}^d$

$$m^n(B) = m_0((X^n)^{-1}(B)) = m_0(\{x \in \mathbb{R}^d : X^n(x) \in B\})\,,$$

or equivalently

$$\langle m^n, \phi \rangle = \langle m_0, \phi(X^n(\cdot)) \rangle \qquad \text{for any } \phi \in C^0_0(\mathbb{R}^d).$$

Above, the dependence of X^n on x and ε and consequently of m^n on ε has been skipped to simplify the notations, but it will be made explicit when necessary below.

It is worth noticing that it is not possible to define the semi-discrete flow (3.7) with ∇u^{n+1} instead of the gradient of the regularized u^{n+1}_ε, since we need X^n to be defined for all $(x, n) \in Q_h$. Moreover, it is also not possible to use an explicit forward Euler scheme to define the trajectories X^n, because the discrete (OSL) condition (3.9) does not provide the necessary equicontinuity of these trajectories, in contrast with the implicit one (3.7), (see Lemma 3.6 and Remark 3.10 below).

The following Lemma gives us the existence, uniqueness and the regularity properties of X^n necessary to the well posedness of m^n and to pass to the limit as h and ε go to 0. Owing to the implicit nature of the Euler scheme (3.7), we have to assume an upper bound on the space-time step h.

Lemma 3.6 (Properties of X^n)**.** *Let $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ and assume ($\mathbf{H}_1$), ($\mathbf{H}'_2$), ($\mathbf{H}_3$), ($\mathbf{H}_4$) and ($\mathbf{A}_1$). Then, if in addition $a \in (C^0(\mathbb{R}^d \times \mathbb{R}^d))^d$, the ($OSL_h$) condition*

$$(a(x, \nabla u^n_\varepsilon(x)) - a(y, \nabla u^n_\varepsilon(y))) \cdot (x - y) \leq C|x - y|^2, \quad x, y \in \mathbb{R}^d\,, n = 0, \cdots, N\,, \tag{3.9}$$

is satisfied, with constant C independent of ε and h, and $C\,h < 1$, the solution of (3.7) *is univocally defined. Moreover, the flow $(n, x) \mapsto X^n(x)$ is locally bounded and Lipschitz continuous w.r.t. $(x, n) \in Q_h$, uniformly in ε.*

Proof. The existence of X^{n+1} given X^n, is insured by the Brouwer fixed point theorem, the map $y \mapsto X^n + h\,a(y, \nabla u_\varepsilon^{n+1}(y))$ being continuous and bounded. The uniqueness of X^{n+1} and the Lipschitz continuity w.r.t. x follow both from (3.9) and the upper bound on h. Indeed, for X^{n+1} and Y^{n+1} defined by (3.7) we have

$$
\begin{aligned}
|X^{n+1} - Y^{n+1}|^2 &= (X^{n+1} - Y^{n+1}) \cdot (X^n - Y^n) \\
&\quad + h\,(X^{n+1} - Y^{n+1}) \cdot \big(a(X^{n+1}, \nabla u_\varepsilon^{n+1}(X^{n+1})) - a(Y^{n+1}, \nabla u_\varepsilon^{n+1}(Y^{n+1}))\big) \\
&\quad \le |X^{n+1} - Y^{n+1}||X^n - Y^n| + C\,h\,|X^{n+1} - Y^{n+1}|^2\,, \qquad (3.10)
\end{aligned}
$$

i.e.

$$|X^{n+1} - Y^{n+1}| \le \left(1 + \frac{C\,h}{1 - C\,h}\right) |X^n - Y^n|\,.$$

Taking $X^n = Y^n$ we get the uniqueness, while iterating over n we get for two starting points $x,\ y \in \mathbb{R}^d$

$$|X^n(x) - X^n(y)| \le \left(1 + \frac{C\,h}{1 - C\,h}\right)^n |x - y|\,.$$

Therefore, for $\delta \in (0, 1)$ s.t. $C\,h \le 1 - \delta$, we have

$$|X^n(x) - X^n(y)| \le \exp(n\,C\,h\,\delta^{-1})|x - y| \le \exp(CT\,\delta^{-1})|x - y|\,, \quad n = 0, \cdots, N\,.$$

Finally, for any $x \in \mathbb{R}^d$ it holds true that

$$|X^n(x) - X^m(x)| \le h|n - m| \sup_{x\in\mathbb{R}^d} \sup_{p\in\overline{B}(0,C_1)} |a(x, p)|\,,$$

where C_1 is given in (3.6), and we obtain the Lipschitz continuity w.r.t. n. The above estimate gives us also that the image by X^n of $B(0, R)$ is contained in the ball of radius $(R + T \sup_{x\in\mathbb{R}^d} \sup_{p\in\overline{B}(0,C_1)} |a(x, p)|)$. Therefore, X^n is locally bounded uniformly in ε and n and the Lemma is proved. □

Lemma 3.7 (Properties of m^n)**.** *Let $m_0 \in \mathcal{M}_1(\mathbb{R}^d)$. Under the same hypothesis as in Lemma 3.6, the approximated solution m^n in* (3.8) *is well defined in $\mathcal{M}_1(\mathbb{R}^d)$. Moreover, m^n satisfies the same properties (i)-(v) as the exact measure solution m, uniformly in n.*

Proof. By Lemma 3.6, the map $x \mapsto X^n(x)$ is uniquely defined and continuous over $\mathbb{R}^d$. Moreover, it is onto from $\mathbb{R}^d$ to $\mathbb{R}^d$. Therefore, the approximated solution m^n is well defined in $\mathcal{M}_1(\mathbb{R}^d)$. Furthermore, it is easy to see that m^n satisfies all the properties (i)-(v) of the exact measure solution m. Concerning (v), it follows from the local and uniform in n boundedness of X^n proved above (see Lemma 3.1 in [16]). □

3.3 The convergence

We can finally discuss the convergence of the semi-discrete scheme (3.3)-(3.7)-(3.8) to the solution of the continuous problem (1.1)-(1.2)-(1.3). The key tools to prove the convergence result have been given in [16], where the authors analyzed the stability of the measure solution with respect to perturbations of the vector field a. The stability of the Filippov characteristics is of course the basic stone. Later, these tools have been adapted in [2] to analyze the stability of the viscosity-measure solution of (1.1)-(1.2)-(1.3) with $a(x,p)=p$, with respect to the vanishing viscosity perturbation of the (HJ) equation.

To discuss the convergence, we define the piecewise constant w.r.t. time approximated solution

$$u_h(x,t)=u^{[t/h]}(x)\,,\quad (x,t)\in Q_T\,,\qquad m_h^\varepsilon(t)=m^{[t/h]}\,,\quad t\in[0,T]\,,$$

and $u_h^\varepsilon=u_h*\rho_\varepsilon$. Moreover, we also need to define the following time continuous trajectories by linear interpolation of $X^n(x)$ and $X^{n+1}(x)$, for any $x\in\mathbb{R}^d$,

$$X_h^\varepsilon(t;x)=X^n(x)+(t-t^n)\,a(X^{n+1}(x),\nabla u_\varepsilon^{n+1}(X^{n+1}(x)))\,,\quad t\in[t^n,t^{n+1}]\,,\quad n=0,\cdots,N\,.\tag{3.11}$$

Note that (3.11) gives us time continuous trajectories X_h^ε with the same regularity as X^n, i.e. X_h^ε is locally bounded and Lipschitz continuous w.r.t. $(t,x)\in[0,T]\times\mathbb{R}^d$, uniformly in ε and h.

Theorem 3.8 (Convergence)**.** *Let $u_0\in(W^{1,\infty}\cap BUC)(\mathbb{R}^d)$ be semiconcave, $m_0\in\mathcal{M}_1(\mathbb{R}^d)$ and assume $(\mathbf{H}_1)$, $(\mathbf{H}_2')$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$ and $(\mathbf{A}_1)$. Assume in addition that $a\in(C^0(\mathbb{R}^d\times\mathbb{R}^d))^d$, u_ε^n satisfies the (OSL_h) condition* (3.9) *and $Ch<1$. Let $\varepsilon=\varepsilon(h)$ be s.t. $\varepsilon(h)\to0$ as $h\to0$. Then, as $h\to0$,*

- *(i) $u_h\to u$ in $L^\infty(Q_T)$ and $\nabla u_h^\varepsilon(x,t)\to\nabla u(x,t)$ at any point $(x,t)\in Q_T$ of differentiability of u, where u is the unique viscosity solution of* (1.1)*;*
- *(ii) X_h^ε converges locally uniformly in Q_T toward the unique Filippov characteristic X associated to the vector field $a(\cdot,\nabla u)$;*
- *(iii) $m_h^\varepsilon\rightharpoonup m$ in $C^0([0,T];\mathcal{M}_1(\mathbb{R}^d)\,w-*)$, where m is the unique measure solution of* (1.2).

Proof. Following standard results of the viscosity solution theory (see for instance [18]), it can be proved that there exists a constant C independent of h, s.t.

$$\|u^n-u(t^n)\|_{L^\infty(\mathbb{R}^d)}\le C\,h^{1/2}\,,\qquad n=0,\cdots,N\,.\tag{3.12}$$

Next, let $(x,t)\in Q_T$ be a point of differentiability of u. For $n=\left[\frac{t}{h}\right]$, by Taylor expansion and the semiconcavity of u_ε^n, we have that

$$u_\varepsilon^n(y)-u_\varepsilon^n(x)-\nabla u_\varepsilon^n(x)\cdot(y-x)\le(C_{conc}^{u_0}+T\,C_{conc}^{H^*})|y-x|^2\,,\quad\forall\,y\in\mathbb{R}^d\,.$$

Therefore, since $\nabla u_\varepsilon^n(x) \to p$ as $h \to 0$, by the uniform bound of ∇u_ε^n, the previous inequality and the convergence of $u_\varepsilon^n(t)$ toward $u(t)$, gives us

$$u(y,t) - u(x,t) - p \cdot (y-x) \le (C_{conc}^{u_0} + T\, C_{conc}^{H^*})(t)|y-x|^2\,,$$

i.e. p belongs to the subdifferential of u at x. Being u differentiable at (x,t), $p = \nabla u(x,t)$ and we have obtained the proof of (i).

We now prove the second part of the statement. From Lemma 3.6 it follows immediately that, up to a subsequence, X_h^ε converges uniformly on every compact set of Q_T to a continuous function Y, as $h \to 0$. Next, since by (i) and the (OSL$_h$) condition (3.9), the (OSL) condition (2.3) is also satisfied, the Filippov characteristic X associated to the vector field $a(\cdot, \nabla u)$ is unique for any $x \in \mathbb{R}^d$. In order to prove that $Y = X$, let us define $\Delta_h^\varepsilon(t;x) := X_h^\varepsilon(t;x) - X(t;x)$. We are going to prove that $\Delta_h^\varepsilon(t;x) \to 0$ as $h \to 0$. Indeed, we have

$$\frac{1}{2}\partial_t |\Delta_h^\varepsilon(t;x)|^2 = \partial_t X_h^\varepsilon(t;x) \cdot \Delta_h^\varepsilon(t;x) - \partial_t X(t;x) \cdot \Delta_h^\varepsilon(t;x)\,. \tag{3.13}$$

By the Definition 2.2 of Filippov characteristics it holds true, for all $x \in \mathbb{R}^d$, a.e. $t \in [0,T]$ and all $r > 0$, that

$$\partial_t X(t;x) \cdot \Delta_h^\varepsilon(t;x) \ge \operatorname*{ess\,inf}_{y \in B(X(t;x),r)} a(y, \nabla u(y,t)) \cdot \Delta_h^\varepsilon(t;x)\,.$$

Hence, for all $\delta > 0$, there exists $\overline{x} = \overline{x}(\delta) \in B(X(t;x), \delta)$ point of differentiability of u, s.t.

$$\partial_t X(t;x) \cdot \Delta_h^\varepsilon(t;x) \ge a(\overline{x}, \nabla u(\overline{x},t)) \cdot \Delta_h^\varepsilon(t;x) - \delta\,. \tag{3.14}$$

Plugging (3.14) into (3.13) and using the definition (3.11) of X_h^ε, we obtain for $n = \left[\frac{t}{h}\right] - 1$,

$$\begin{array}{rcl} \frac{1}{2}\partial_t |\Delta_h^\varepsilon(t;x)|^2 & \le & a(X^{n+1}(x), \nabla u_\varepsilon^{n+1}(X^{n+1}(x))) \cdot \Delta_h^\varepsilon(t;x) - a(\overline{x}, \nabla u(\overline{x},t)) \cdot \Delta_h^\varepsilon(t;x) + \delta \\ & = & \big(a(X^{n+1}(x), \nabla u_\varepsilon^{n+1}(X^{n+1}(x))) - a(\overline{x}, \nabla u_\varepsilon^{n+1}(\overline{x}))\big) \cdot \Delta_h^\varepsilon(t;x) \\ & & \quad + \big(a(\overline{x}, \nabla u_\varepsilon^{n+1}(\overline{x})) - a(\overline{x}, \nabla u(\overline{x},t))\big) \cdot \Delta_h^\varepsilon(t;x) + \delta := I_1 + I_2 + \delta\,. \end{array} \tag{3.15}$$

In order to estimate I_1, we decompose $\Delta_h^\varepsilon(t;x)$ as

$$\Delta_h^\varepsilon(t;x) = (X_h^\varepsilon(t;x) - X^{n+1}(x)) + (X^{n+1}(x) - \overline{x}) + (\overline{x} - X(t;x))\,,$$

we note $\mathcal{A} := \sup_{x \in \mathbb{R}^d} \sup_{p \in \overline{B}(0,C_1)} |a(x,p)|$, where C_1 is given in (3.6), and we make use of the (OSL$_h$) condition (3.9) to get

$$I_1 \le 2\mathcal{A}\,|X_h^\varepsilon(t;x) - X^{n+1}(x)| + C\,|X^{n+1}(x) - \overline{x}|^2 + 2\mathcal{A}\,\delta \le 2\mathcal{A}^2\,h + C\,|X^{n+1}(x) - \overline{x}|^2 + 2\mathcal{A}\delta\,. \tag{3.16}$$

¿From (3.16), (3.15) and the uniform boundedness of $\Delta_h^\varepsilon(t;x)$ on every compact subset of Q_T, we have obtained

$$\frac{1}{2}\partial_t |\Delta_h^\varepsilon(t;x)|^2 \le 2\mathcal{A}^2\,h + C\,|X^{n+1}(x) - \overline{x}|^2 + 2\mathcal{A}\delta + C|a(\overline{x}, \nabla u_\varepsilon^{n+1}(\overline{x})) - a(\overline{x}, \nabla u(\overline{x},t))| + \delta\,.$$

Using the convergence results obtained above and passing to the limit into the previous inequality as $h \to 0$ and $\delta \to 0$, we get that

$$\frac{1}{2}\partial_t |Y(t;x) - X(t;x)| \leq C\,|Y(t;x) - X(t;x)|^2\,,$$

i.e. $Y(t;\cdot) = X(t;\cdot)$ in $L^2_{loc}(\mathbb{R}^d)$ a.e. $t \in [0,T]$. Finally, from the continuity of Y and X we deduce that $Y = X$. By the uniqueness of X and the uniform boundedness of X^ε_h, we have that all the sequence X^ε_h converge.

It remains to prove the convergence of m^ε_h. This can be obtained exactly as in Proposition 3.1 in [16] and we skip the proof. We just underline that the strong convergence of the trajectories X^ε_h, their time equicontinuity and the uniform compactness at infinity of m^ε_h (property (v)) are the key tools to prove that the sequence $\langle m^\varepsilon_h(t), \phi\rangle$ is convergent and equicontinuous on $[0,T]$, for any $\phi \in C^0_0(\mathbb{R}^d)$. $\square$

Proceeding as in (2.5), one can obtain the (OSL_h) condition (3.9) for special a. This is summarized in the following Corollary which is a straightforward consequence of Theorem 3.8. It is worth noticing that the Hamiltonians (1.5) and (1.6) enter in the framework of this Corollary.

Corollary 3.9. *Assume the same hypothesis of Theorem 3.8, except the (OSL_h) condition* (3.9). *If in addition, a is differentiable w.r.t. p, satisfies a one sided Lipschitz condition w.r.t. x locally uniformly in p and either $D_p a(x,p) = I$ or $d = 1$ and $\partial_p a(x,p)$ is nonnegative and upper bounded, then the same conclusions as in Theorem 3.8 hold true.*

Remark 3.10 (The explicit Euler scheme) The explicit Euler scheme

$$X^{n+1} = X^n + h\, a(X^n, \nabla u^n_\varepsilon(X^n))\,, \tag{3.17}$$

does not provide a family of equicontinuous characteristics, under the (OSL_h) condition (3.9). This is a difficulty naturally intrinsic to the (OSL) condition that allows discontinuity of compressive type only, while the map $x \mapsto X^n(x)$ given by (3.17) can be expansive. Indeed, we have

$$\begin{aligned}
|X^{n+1} - Y^{n+1}|^2 &= (X^{n+1} - Y^{n+1}) \cdot (X^n - Y^n) \\
&\quad + h\,(X^{n+1} - Y^{n+1}) \cdot (a(X^n, \nabla u^n_\varepsilon(X^n)) - a(Y^n, \nabla u^n_\varepsilon(Y^n))) \\
&\quad \leq \frac{1}{2}|X^{n+1} - Y^{n+1}|^2 + \frac{1}{2}|X^n - Y^n|^2 + C\,h\,|X^n - Y^n|^2 + h^2 |a(X^n, \nabla u^n_\varepsilon(X^n)) - a(Y^n, \nabla u^n_\varepsilon(Y^n))|^2\,,
\end{aligned}$$

i.e., with the same constant $\mathcal{A}$ as in Theorem 3.8,

$$|X^{n+1} - Y^{n+1}|^2 \leq (1 + 2Ch)|X^n - Y^n|^2 + 8\,\mathcal{A}^2\,h^2$$

and iterating over n : $|X^n(x) - X^n(y)|^2 \leq e^{2CT}\left[|x-y|^2 + 8\,\mathcal{A}^2\,T\,h\right]$.

4 The fully discrete semi-lagrangian scheme

In this section we introduce a finite element discretization of (3.3) and an approximation of (3.8) by a bounded discrete measure, yielding a fully discrete scheme for (1.1)-(1.2)-(1.3).

For an arbitrarily fixed space step $k > 0$, we consider the regular uniform grid of $\mathbb{R}^d$ given by $\mathcal{X}^k := \{x_i = ik\,,\ i \in \mathbb{Z}^d\}$. Let $\mathcal{T}^k = \{S_j^k\}_{j\in\mathcal{J}^k}$ be the associated collection of non-degenerate, pairwise disjoint and uniform simplices having as vertices lattice points $x_i \in \mathcal{X}^k$ and covering $\mathbb{R}^d$, (S_j^k are triangles in dimension 2 and tetrahedra in dimension 3). We denote also by

$$W^k = \{w \in C(\mathbb{R}^d) : w \text{ is linear on } S_j^k,\ j \in \mathcal{J}^k\},$$

the space of continuous piecewise linear functions on $\mathcal{T}^k$. Then, each $w \in W^k$ can be expressed as

$$w(x) = \sum_{i\in\mathbb{Z}^d} \beta_i^k(x) w(x_i), \tag{4.1}$$

for basis functions $\beta_i^k \in W^k$ satisfying $\beta_i^k(x_j) = \delta_{ij}$ for $i, j \in \mathbb{Z}^d$. It immediately follows that any β_i^k has compact support, $0 \le \beta_i^k(x) \le 1$, $\sum_{i\in\mathbb{Z}^d} \beta_i^k(x) = 1$ and at any $x \in \mathbb{R}^d$ at most $(d+1)$ functions β_i^k are non-zero. In the sequel, the interpolation operator defined by (4.1) will be denoted P_k.

The fully discrete approximation of (3.3) based on the above space discretization is naturally given by

$$\begin{cases} u_k^n(x) = \sum_{i\in\mathbb{Z}^d} \beta_i^k(x)\, u_{k,i}^n\,, & n = 0, \cdots, N\,, \\[2ex] u_{k,i}^{n+1} = \inf_{\xi\in\mathbb{R}^d} \{u_k^n(x_i - \xi h) + h\, H^*(x_i, t^n, \xi)\}\ , & i \in \mathbb{Z}^d\,, \end{cases} \tag{4.2}$$

where $u_{k,i}^0 = u_0(x_i)$. It is obvious that, thanks to the continuous piecewise linear interpolation of the discrete approximation $(u_{k,i}^n)_{i\in\mathbb{Z}^d}$ on the lattice at any time step, the continuous function $u_k^n \in W^k$ shares the properties of the semi-discrete approximation u^n given in Lemma 3.3. Therefore, the equivalent of Lemma 3.3 for the fully discrete approximation u_k^n will be skipped here. On the other hand, the semiconcavity of u_k^n, given a semiconcave initial data u_0, is not straightforward. Therefore, we shall give the equivalent of Lemma 3.4 here and the proof in the Appendix.

Lemma 4.1 (Semiconcavity)**.** *Let $u_0 \in W^{1,\infty}(\mathbb{R}^d)$ be semiconcave. Then, under hypothesis* ($\mathbf{H}_1$), ($\mathbf{H}_2'$), ($\mathbf{H}_3$), ($\mathbf{H}_4$) *and* ($\mathbf{H}_5$), *u_k^n is discretely semiconcave for all $n = 0, \cdots N$, i.e.*

$$u_k^n(x + x_j) - 2\, u_k^n(x) + u_k^n(x - x_j) \le (C_{conc}^{u_0} + T\, C_{conc}^{H^*})|x_j|^2\,, \quad x \in \mathbb{R}^d\,,\ j \in \mathbb{Z}^d\,, \tag{4.3}$$

and weakly semiconcave on $\mathbb{R}^d$, i.e.

$$u_k^n(x + y) - 2\, u_k^n(x) + u_k^n(x - y) \le (C_{conc}^{u_0} + T\, C_{conc}^{H^*}) \left[|y|^2 + \frac{k^2}{2}(E(x + y) + E(x - y))\right]\ , \quad x, y \in \mathbb{R}^d\,, \tag{4.4}$$

where $E(x)$ is a nonnegative, continuous and bounded function vanishing over $\mathcal{X}^k$.

Remark 4.2 We have chosen to consider a uniform grid $\mathcal{X}^k$, i.e. a uniform space step k in any axial direction, only for simplicity of notations. It is obvious that Lemma 4.1 still holds true if one chooses different space steps for each direction of the grid, but with an additional non-degeneracy condition. On the other hand, it seems difficult to obtain the discrete-semiconcavity of u_k^n if the regular lattice $\mathcal{X}^k$ is replaced with a general non-degenerate triangulation of $\mathbb{R}^d$. Moreover, this key property strongly depends on the continuous piecewise linear interpolation (4.1). Therefore, it is not possible to use a nonlinear interpolation operator in order to preserve the semiconcavity of $|x|^2$ and to obtain the semiconcavity of u_k^n over $\mathbb{R}^d$ from (4.3).

We now turn to the approximation of (3.8). A definition of space continuous trajectories is however always necessary. Therefore, let $u_{k,\varepsilon}^n = u_k^n * \rho_\varepsilon$ and set

$$\begin{cases} X_k^{n+1} = X_k^n + h\, a(X_k^{n+1}, \nabla u_{k,\varepsilon}^{n+1}(X_k^{n+1}))\,, & n = 0, \cdots, N-1, \\ X_k^0 = x\,, \qquad x \in \mathbb{R}^d\,. \end{cases} \tag{4.5}$$

Again, the existence of X_k^{n+1} given X_k^n is due to the Brouwer fixed point theorem applied to the map $y \mapsto X_k^n + h\, a(y, \nabla u_{k,\varepsilon}^{n+1}(y))$. However, the same argument as in Lemma 3.6 giving the uniqueness of X_k^{n+1} cannot be reproduced here. Indeed, due to the weak semiconcavity (4.4), $u_{k,\varepsilon}^n$ is only weak semiconcave in general. Therefore, we are allowed to assume only the following weak (OSL_h^k) condition,

$$\big(a(x, \nabla u_{k,\varepsilon}^n(x)) - a(y, \nabla u_{k,\varepsilon}^n(y))\big) \cdot (x-y) \leq C'\,|x-y|^2, \quad x, y \in \mathbb{R}^d\,, \quad |x-y| \geq k\,, \tag{4.6}$$

for $n = 0, \cdots, N$ and a constant C' independent of h, k and ε.

In order to obtain the well posedeness of the implicit Euler scheme (4.5), we assume

($\mathbf{A}_2$) a locally Lipschitz w.r.t. p uniformly in x and one sided Lipschitz continuous w.r.t. x locally uniformly in p.

Then, combining hypothesis ($\mathbf{A}_2$) with the Lipschitz property of u_k^n, there exists a constant C'', independent of h, k and ε, such that if h is small enough, i.e.

$$C''\, h\, \varepsilon^{-1} < 1\,, \tag{4.7}$$

the previous map is contracting, X_k^{n+1} is unique and the flow $(n, x) \mapsto X_k^n(x)$ is locally uniformly bounded and Lipschitz continuous w.r.t. $(x, n) \in Q_h$, uniformly in k.

Remark 4.3 Condition (4.6) gives us nonetheless the interesting property that the characteristics X_k^n do not move much away from each other. Indeed, for X_k^{n+1} and Y_k^{n+1} defined by (4.5) and such that $|X_k^{n+1} - Y_k^{n+1}| \geq k$, proceeding as in (3.10), we obtain

$$|X_k^{n+1} - Y_k^{n+1}| \leq (1 + C'\, h\, \delta^{-1})|X_k^n - Y_k^n|\,.$$

Therefore,

$$|X_k^{n+1} - Y_k^{n+1}| \leq (1 + C'\, h\, \delta^{-1}) \max\{|X_k^n - Y_k^n|,\, k\}\,,$$

and iterating over n,

$$|X_k^n(x) - X_k^n(y)| \leq \exp(C' \, T \, \delta^{-1}) \max\{|x-y|, \, k\} \, . \tag{4.8}$$

Next, let δ_i be the Dirac measure concentrated on the lattice points x_i. Let $m_k^0 := \sum_{i \in \mathbb{Z}^d} m_{k,i}^0 \, \delta_i$ be an approximation of the initial measure m_0 in the space of discrete bounded measures, for the $\mathcal{M}_1(\mathbb{R}^d) \, w - *$ topology, conserving the mass and that tends to 0 at infinity uniformly w.r.t. k sufficiently small, i.e. for all $\varepsilon > 0$ there exist $R > 0$ and $K > 0$ s.t.

$$|m_k^0|(\mathbb{R}^d \setminus B_R(0)) < \varepsilon \qquad \forall \, k < K \, . \tag{4.9}$$

For example, one can consider $m_{k,i}^0 := m_0(A_i^k)$, with $(A_i^k)_{i \in \mathbb{Z}^d}$ a partition of $\mathbb{R}^d$ such that $x_j \in A_i^k$ iff $i = j$. Then, the mass is obviously conserved and (4.9) is satisfied since for any Borel set B

$$|m_k^0|(B) \leq |m_0| \left(\cup_{i \in I_k} A_i^k \right) \, , \qquad I_k = \{ i \in \mathbb{Z}^d \, : \, x_i \in B \} \, .$$

We define μ_k^n as the image of m_k^0 by means of the flow $X_k^n(\cdot)$. As for m^n, μ_k^n is well defined in $\mathcal{M}_1(\mathbb{R}^d)$, the map $x \mapsto X_k^n(x)$ being uniquely defined, continuous and onto from $\mathbb{R}^d$ to $\mathbb{R}^d$. μ_k^n satisfies also all the properties (i)-(v) with respect to m_k^0, for any n. However, μ_k^n is not a discrete bounded measure. Hence, following the classical procedure of the finite element approximation, we observe first that to determine μ_k^n, it is sufficient to test the equation

$$\langle \mu_k^n, \phi \rangle = \langle m_k^0, \phi(X_k^n(\cdot)) \rangle \, , \tag{4.10}$$

against any $\phi \in C_c^0(\mathbb{R}^d)$. Indeed, $C_c^0(\mathbb{R}^d)$ is dense in $C_0^0(\mathbb{R}^d)$ and μ_k^n tends to 0 at infinity, uniformly in t and in k sufficiently large owing to (4.9). Moreover, since any function in $C_c^0(\mathbb{R}^d)$ can be approximated uniformly by a function $w \in W^k$ with compact support, (4.10) becomes for such test functions

$$\langle \mu_k^n, w \rangle = \sum_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} m_{k,i}^0 \, \beta_j^k(X_k^n(x_i)) \, w(x_j) = \sum_{j \in \mathbb{Z}^d} m_{k,j}^n \, w(x_j) \, , \tag{4.11}$$

with $m_{k,j}^n := \sum_{i \in \mathbb{Z}^d} m_{k,i}^0 \, \beta_j^k(X_k^n(x_i))$. Let us observe that the last identity in (4.11) holds since the series $\sum_{i \in \mathbb{Z}^d} m_{k,i}^0$ is absolutely convergent and for the same reason $m_{k,j}^n$ is well defined. Finally, (4.11) leads naturally to the following definition of the discrete bounded measure approximating μ_k^n

$$m_k^n := \sum_{i \in \mathbb{Z}^d} m_{k,i}^n \, \delta_i \, . \tag{4.12}$$

It immediately follows that m_k^n conserves the mass, i.e.

$$\sum_{i \in \mathbb{Z}^d} m_{k,i}^n = \sum_{i \in \mathbb{Z}^d} m_{k,i}^0 = m_0(\mathbb{R}^d) \, , \qquad n = 0, \ldots, N \, .$$

Furthermore, property (4.9) is also satisfied by m_k^n uniformly in $n = 0, \ldots, N$, by the definition of the coefficients $m_{k,i}^n$ and (4.8).

With the above definitions, set $U_k^n = (u_{k,i}^n)_{i\in\mathbb{Z}^d}$ and $M_k^n = (m_{k,i}^n)_{i\in\mathbb{Z}^d}$. The fully discrete scheme (4.2), (4.12), reads

$$\begin{cases} U_k^{n+1} = \inf_{\xi\in\mathbb{R}^d}\{B^k(\xi)\,U_k^n + h\,L^n(\xi)\}\,, \qquad n = 0,\cdots,N-1\,, \\ M_k^n = \Lambda_k^n\,M_k^0\,, \qquad n = 1,\cdots,N\,, \end{cases} \tag{4.13}$$

where $B^k(\xi) = (\beta_i^k(x_j - h\xi))_{i,j\in\mathbb{Z}^d}$, $L^n(\xi) = (H^*(x_i, t^n, \xi))_{i\in\mathbb{Z}^d}$ and $\Lambda_k^n = (\beta_i^k(X_k^n(x_j)))_{i,j\in\mathbb{Z}^d}$. It is worth noticing that B^k is a stochastic matrix, while Λ_k^n is the transpose of a stochastic matrix.

Defining again the following time piecewise constant approximations

$$u_{h,k}(x,t) = u_k^{[t/h]}(x)\,, \quad (x,t)\in Q_T\,, \qquad m_{h,k}^\varepsilon(t) = m_k^{[t/h]}\,, \quad t\in[0,T]\,, \tag{4.14}$$

as well as $u_{h,k}^\varepsilon = u_{h,k} * \rho_\varepsilon$ and the time linear interpolated trajectories $X_{h,k}^\varepsilon$ exactly as in (3.11), we can prove the convergence of the fully discrete scheme (4.13).

Theorem 4.4 (Convergence)**.** *Let $u_0 \in (W^{1,\infty}\cap BUC)(\mathbb{R}^d)$ be semiconcave and $m_0 \in \mathcal{M}_1(\mathbb{R}^d)$. Assume hypothesis $(\mathbf{H}_1)$, $(\mathbf{H}_2')$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$, $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$. Assume in addition that the (OSL_h^k) condition (4.6) is satisfied with h sufficiently small such that $C'\,h < 1$ and that (4.7) is satisfied with $\varepsilon = h^\alpha$, $\alpha\in(0,1)$. Then, as $\frac{k}{h^{1+\alpha}} \to 0$,*

(i) $u_{h,k} \to u$ in $L^\infty(Q_T)$ and $\nabla u_{h,k}^\varepsilon(x,t) \to \nabla u(x,t)$ in any point $(x,t)\in Q_T$ of differentiability of u, where u is the unique viscosity solution of (1.1);

(ii) $X_{h,k}^\varepsilon$ converges locally uniformly in Q_T toward the unique Filippov characteristic X associated to the vector field $a(\cdot,\nabla u)$;

(iii) $m_{h,k}^\varepsilon \rightharpoonup m$ in $C^0([0,T];\mathcal{M}_1(\mathbb{R}^d)\,w-*)$, where m is the unique measure solution of (1.2).

Proof. We shall prove that there exists a constant C independent of h and k, such that

$$\|u_k^n - u^n\|_{L^\infty(\mathbb{R}^d)} \le C(n+1)k\,, \qquad n = 0,\ldots,N\,. \tag{4.15}$$

As a consequence, by the time regularity of u and by estimate (3.12), it follows that

$$\|u_{h,k}(t) - u(t)\|_{L^\infty(\mathbb{R}^d)} \le C(h + \frac{k}{h} + h^{1/2})\,, \qquad \forall\, t\in[0,T]\,.$$

Estimate (4.15) is obvious for $n = 0$, being $u_k^0 = P_k u_0$ and $u^0 = u_0$. Let us suppose that it holds true for a given n. Then, for any argument $\alpha^n(x_i)\in A^n(x_i)$, we have by (3.3), (3.4) and (4.2)

$$u_{k,i}^{n+1} - u^{n+1}(x_i) \le u_k^n(x_i - h\,\alpha^n(x_i)) - u^n(x_i - h\,\alpha^n(x_i)) \le C(n+1)k\,.$$

Exchanging the roles of $u_{k,i}^{n+1}$ and $u^{n+1}(x_i)$ we obtain

$$|u_{k,i}^{n+1} - u^{n+1}(x_i)| \le C(n+1)k\,, \quad \forall\, i\in\mathbb{Z}^d\,,$$

so that for any $x \in \mathbb{R}^d$,

$$|u_k^{n+1}(x) - u^{n+1}(x)| \le |u^{n+1}(x) - P_k u^{n+1}(x)| + \sum_{i\in\mathbb{Z}^d} \beta_i^k(x)|u^{n+1}(x_i) - u_{k,i}^{n+1}| \le Ck + C(n+1)k\,.$$

Since the convergence of $\nabla u_{h,k}^\varepsilon$ can be proved exactly as in Theorem 3.8, from the weak-semiconcavity of $u_{k,\varepsilon}^n$, statement (i) is proved.

Next, from (4.15), we have the same estimate for the regularized approximation $u_{k,\varepsilon}^n$ and u_ε^n, yielding

$$\|\nabla u_{k,\varepsilon}^n - \nabla u_\varepsilon^n\|_{L^\infty(\mathbb{R}^d)} \le C\frac{k}{\varepsilon\, h}\,, \qquad n = 0,\ldots,N\,. \tag{4.16}$$

Then, from the (OSL_h^k) condition (4.6), the (OSL_h) condition (3.9) with constant C' follows as $k \to 0$, and the trajectories X^n can be defined as in (3.7), under the assumption $C'h < 1$. Moreover,

$$\begin{aligned}
|X_k^{n+1} - X^{n+1}|^2 &= (X_k^{n+1} - X^{n+1})\cdot(X_k^n - X^n)\\
&\quad + h\,(X_k^{n+1} - X^{n+1})\cdot\Big(a(X_k^{n+1},\nabla u_{k,\varepsilon}^{n+1}(X_k^{n+1})) - a(X^{n+1},\nabla u_\varepsilon^{n+1}(X^{n+1}))\Big)\\
&\le |X_k^{n+1} - X^{n+1}||X_k^n - X^n|\\
&\quad + h\,(X_k^{n+1} - X^{n+1})\cdot\Big(a(X_k^{n+1},\nabla u_{k,\varepsilon}^{n+1}(X_k^{n+1})) - a(X_k^{n+1},\nabla u_\varepsilon^{n+1}(X_k^{n+1}))\Big)\\
&\quad + h\,(X_k^{n+1} - X^{n+1})\cdot\big(a(X_k^{n+1},\nabla u_\varepsilon^{n+1}(X_k^{n+1})) - a(X^{n+1},\nabla u_\varepsilon^{n+1}(X^{n+1}))\big)\\
&\le |X_k^{n+1} - X^{n+1}||X_k^n - X^n| + h\,(\mathrm{Lip}_p a)\,|X_k^{n+1} - X^{n+1}||\nabla u_{k,\varepsilon}^{n+1}(X_k^{n+1}) - \nabla u_\varepsilon^{n+1}(X_k^{n+1})|\\
&\quad + C'\,h|X_k^{n+1} - X^{n+1}|^2\,,
\end{aligned}$$

i.e. using (4.16),

$$|X_k^{n+1} - X^{n+1}| \le (1 + C'\,h\,\delta^{-1})\left(|X_k^n - X^n| + \tilde{C}\,\frac{k}{\varepsilon}\right)\,,$$

for $\delta \in (0,1)$ s.t. $C'\,h \le 1-\delta$. Iterating over n, we obtain for any $x \in \mathbb{R}^d$

$$|X_k^n(x) - X^n(x)| \le \tilde{C}\,\frac{k}{\varepsilon}\,\sum_{i=1}^n (1 + C'\,h\,\delta^{-1})^i \le C\,\frac{k}{\varepsilon\, h}\,. \tag{4.17}$$

Combining (4.17) with statement (ii) in Theorem 3.8, we have proved statement (ii).

Finally, to obtain the convergence of $m_{h,k}^\varepsilon$ towards m, it is enough to prove that, as $k\,, h \to 0$, $(m_{h,k}^\varepsilon - m_h^\varepsilon) \rightharpoonup 0$ in $C^0([0,T];\mathcal{M}_1(\mathbb{R}^d)\,w{-}*)$. The latter combined with statement (iii) in Theorem 3.8, gives us the claim.

Again, using the compactness at infinity of $(m_{h,k}^\varepsilon - m_h^\varepsilon)$, uniformly in $\varepsilon\,, h\,, t$ and k sufficiently large, it is possible to consider only test functions $\phi \in C_c^0(\mathbb{R}^d)$. Then, for $n = \left[\frac{t}{h}\right]$, we have

$$\begin{aligned}
\langle m_{h,k}^\varepsilon(t) - m_h^\varepsilon(t), \phi\rangle &= \langle m_k^n - m^n, \phi\rangle = \langle m_k^n - m^n, \phi - P_k\phi\rangle + \langle m_k^n - m^n, P_k\phi\rangle\\
&= \sum_{i\in\mathbb{Z}^d} m_{k,i}^n(\phi(x_i) - P_k\phi(x_i)) - \langle m_0, \phi(X^n(\cdot)) - P_k\phi(X^n(\cdot))\rangle + \langle m_k^n - m^n, P_k\phi\rangle\\
&:= I_1 + I_2 + I_3\,.
\end{aligned}$$

The term I_1 is obviously equal to 0. The term I_2 goes to 0 as $k\,,h \to 0$ by Lebesgue dominated convergence theorem. Concerning I_3, using (4.11), we have

$$\begin{aligned} I_3 &= \langle \mu^n_k, P_k\phi\rangle - \langle m^n, P_k\phi\rangle = \langle m^0_k, P_k\phi(X^n_k(\cdot))\rangle - \langle m_0, P_k\phi(X^n(\cdot))\rangle \\ &= \langle m^0_k, P_k\phi(X^n_k(\cdot)) - P_k\phi(X^n(\cdot))\rangle + \langle m^0_k - m_0, P_k\phi(X^n(\cdot))\rangle \\ &= \sum_{i\in\mathbb{Z}^d} m^0_{k,i}\,(P_k\phi(X^n_k(x_i)) - P_k\phi(X^n(x_i))) + \langle m^0_k - m_0, P_k\phi(X^n(\cdot))\rangle := I_3' + I_3''\,. \end{aligned}$$

By (4.17), the uniform continuity of $P_k\phi$ and the uniform boundedness of m^0_k, $I_3' \to 0$ as $k\,,h \to 0$. The same holds true for I_3'' since $m^0_k \rightharpoonup m_0$ in $\mathcal{M}_1(\mathbb{R}^d)\,w-*$ as $k\to 0$. To conclude, the sequence $\langle m^\varepsilon_{h,k}(t) - m^\varepsilon_h(t), \phi\rangle$ converging to 0, is also equicontinuous on $[0,T]$ since $\langle m^\varepsilon_h(t),\phi\rangle$ is equicontinuous (see Theorem 3.8) and

$$|\langle m^\varepsilon_{h,k}(t_1) - m^\varepsilon_{h,k}(t_2), \phi\rangle| \le \sum_{i\in\mathbb{Z}^d} |m^{n_1}_{k,i} - m^{n_2}_{k,i}|\,|\phi(x_i)|\,.$$

Using the uniform continuity of the basis functions, the time equicontinuity of the characteristics X^n and (4.17) in the estimate below

$$\begin{aligned} |m^{n_1}_{k,i} - m^{n_2}_{k,i}| \le \sum_{j\in\mathbb{Z}^d} |m^0_{k,j}|\,[\,|\beta^k_i(X^{n_1}_k(x_j)) - \beta^k_i(X^{n_1}(x_j))| + |\beta^k_i(X^{n_1}(x_j)) - \beta^k_i(X^{n_2}(x_j))| + \\ |\beta^k_i(X^{n_2}(x_j)) - \beta^k_i(X^{n_2}_k(x_j))|\,]\,, \end{aligned}$$

we obtain the claim. ◻

5 Appendix

Proof of Lemma 3.2. (Properties of H^*). If H satisfies $(\mathbf{H}_4)$-(i), we need only to prove that $H^*(\cdot,\cdot,0) \in L^\infty(Q_T)$, the other claims being straightforward. Then, let us define $A = \{p \in \mathbb{R}^d \ :\ H(x,t,p) \le M\,,\ \forall\,(x,t)\in Q_T\}$, where $M \equiv \sup_{Q_T}|H(x,t,0)|$. A is a non empty set of $\mathbb{R}^d$ and it is bounded since H growths linearly for $|p|\to+\infty$ and

$$\frac{H(x,t,p)}{|p|} \le \frac{M}{|p|}\,,\quad \forall\, p\in A\,.$$

Thus, there exists $R>0$ such that $A \subset B(0,R)$ and

$$H^*(x,t,0) = \sup_{p\in\mathbb{R}^d}\{-H(x,t,p)\} = \max_{p\in\overline{B}(0,R)}\{-H(x,t,p)\} \in L^\infty(Q_T)\,.$$

On the other end, if H satisfies $(\mathbf{H}_4)$-(ii), the uniform superlinearity of H^* is a direct consequence. To prove (3.5), let us denote $A(\xi) = \{p\in\mathbb{R}^d \ :\ p\cdot\xi - H(x,t,p) \ge -M\,,\ \forall\,(x,t)\in Q_T\}$. Then, $A(\xi)$ is again non empty ($0\in A(\xi)$ for all ξ) and bounded whenever $|\xi|\le r$ since

$$\frac{H(x,t,p)}{|p|} \le \frac{p\cdot\xi}{|p|} + \frac{M}{|p|} \le r + \frac{M}{|p|}\,,\quad \forall\, p\in A\,,$$

uniformly in $(x,t) \in Q_T$. Finally, it follows immediately that $H^* \in L^\infty(Q_T \times \overline{B}(0,r))$ and

$$|H^*(x,t,\xi) - H^*(y,t,\xi)| \le \eta(1+R)|x-y|\,, \quad \forall\, x,y \in \mathbb{R}^d\,, \forall\, t \in [0,T]\,, \forall\, \xi \in \overline{B}(0,r)\,.$$

□

Proof of Lemma 4.1. (Semiconcavity of u_k^n). We follow here [8],[9] and [15]. The discrete semiconcavity (4.3) for $x = x_i \in \mathcal{X}^k$ can be proved by induction. Indeed, this is true for u_k^0 by the semiconcavity of u_0. Let $\alpha_{k,i}^n$ be one argument of the infimum in (4.2). Then,

$$u_{k,i+j}^{n+1} - 2\, u_{k,i}^{n+1} + u_{k,i-j}^{n+1} \le u_k^n(x_{i+j} - h\,\alpha_{k,i}^n) - 2u_k^n(x_i - h\,\alpha_{k,i}^n) + u_k^n(x_{i-j} - h\,\alpha_{k,i}^n) + h\, C_{conc}^{H^*}|x_j|^2\,. \quad (5.1)$$

Next, since the point $(x_i - h\,\alpha_{k,i}^n)$ belongs to a simplex S_j^k, it can be written as a convex combination of its vertices, i.e. $(x_i - h\,\alpha_{k,i}^n) = \sum_l \lambda_l x_l$, where $\lambda_l \in [0,1]$ and $\sum_l \lambda_l = 1$. By the regularity of the lattice, the same holds true for $(x_{i\pm j} - h\,\alpha_{k,i}^n) = \sum_l \lambda_l x_{l\pm j}$. Finally, being u_k^n piecewise linear on $\mathcal{T}^k$, (5.1) becomes

$$u_{k,i+j}^{n+1} - 2\, u_{k,i}^{n+1} + u_{k,i-j}^{n+1} \le \sum_l \lambda_l u_k^n(x_{l+j}) - 2\sum_l \lambda_l u_k^n(x_l) + \sum_l \lambda_l u_k^n(x_{l-j}) + h\, C_{conc}^{H^*}|x_j|^2\,,$$

and (4.3) is proved on the lattice $\mathcal{X}^k$. For an arbitrary $x \in \mathbb{R}^d$, we have again as before $x = \sum_l \lambda_l x_l$ and $x \pm x_j = \sum_l \lambda_l x_{l\pm j}$. Hence,

$$u_k^n(x+x_j) - 2\, u_k^n(x) + u_k^n(x-x_j) = \sum_l \lambda_l [u_{k,l+j}^{n+1} - 2\, u_{k,l}^{n+1} + u_{k,l-j}^{n+1}]\,,$$

and (4.3) follows.

Set $C = (C_{conc}^{u_0} + h\, C_{conc}^{H^*})$, inequality (4.3) implies that the continuous piecewise linear interpolation of $(u_{k,i}^n - \frac{C}{2}\,|x_i|^2)_{i\in\mathbb{Z}^d}$ is concave on $\mathcal{X}^k$ for all $n = 0, \cdots N$, and consequently also over $\mathbb{R}^d$. Therefore, for any $x,y \in \mathbb{R}^d$

$$u_k^{n+1}(x+y) - 2u_k^{n+1}(x) + u_k^{n+1}(x-y) \le \frac{C}{2}\left[\sum_{i\in\mathbb{Z}^d} \beta_i^k(x+y)|x_i|^2 - 2\sum_{i\in\mathbb{Z}^d} \beta_i^k(x)|x_i|^2 + \sum_{i\in\mathbb{Z}^d} \beta_i^k(x-y)|x_i|^2\right]\,.$$

Since the piecewise linear interpolation of $|x|^2$ satisfies

$$|x|^2 \le \sum_{i\in\mathbb{Z}^d} \beta_i^k(x)|x_i|^2 \le |x|^2 + k^2\, E(x)\,,$$

with $E(x)$ the approximation error, the weak-semiconcavity (4.4) is also proved. □

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